

The thermodynamic relationship between the charged AdS black holes and the asymptotically flat space-time charged black hole

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Abstract. In this paper, by analyzing the thermodynamic properties of charged AdS black hole and asymptotically flat space-time charged black hole in the vicinity of the critical point, we establish the correspondence between the thermodynamic parameters of asymptotically flat space-time and non-asymptotic flat space-time, based on the equality of black hole horizon in the two space-time. The relationship between the cavity radius (which is introduced in the study of asymptotically flat space-time charged black holes) and the cosmological constant (which is introduced in the study of non-asymptotic flat space-time) is determined. The establishment of the correspondence between the thermodynamics parameters in two space-time is beneficial to the mutual promotion of different time-space black hole research, which is helpful to understand the thermodynamics and quantum properties of black hole in space-time.

1 Introduction

The AdS black hole solution in four-dimensional space-time, which was originally obtained by Hawking and Page (1983) (author?) [1], is an accurate black hole solution of the Einstein equation with negative cosmological constant in asymptotic AdS space time. This solution has the same thermodynamic characteristics as the black hole solution in asymptotic flat space-time, i.e., the black hole entropy is equal to a quarter of the event horizon area, while the corresponding thermodynamics quantity satisfies the law of thermodynamics of black hole. It is well known that, if taken as a thermodynamic system, the asymptotic flat black hole does not meet the requirements of thermodynamic equilibrium stability due to its negative heat capacity. However, compared with the asymptotically flat space-time black hole, the AdS black hole can be in thermodynamic equilibrium and stable state, because the heat capacity of the system is positive when the system parameters take certain values.

Therefore, the investigation of the thermodynamics of AdS black hole, in particular the study of its phase transition, has been extensively discussed in recent works. In the previous literature, the cosmological constants Λ is always taken as an independent thermodynamic parameter (pressure), and the first law of thermodynamics of AdS black hole is also established with $P - V$ terms. For instance, the $P - V$ critical properties of AdS black hole was firstly studied in Ref [2], which found that the phase

transition and critical behavior of RN-AdS black hole are similar to those of the general van der waals-Maxwell system. More specifically, the RN-AdS black hole exhibits the same $P - V$ (Liauid-Gas Phase Transitions) critical phase transition behavior and critical index as van der waals-Maxwell system. The phase transition and critical behavior of various black holes in the extended phase space of AdS have been extensively discussed in the literature [3–20], which showed very similar phase diagrams in different black hole systems.

The asymptotic flat black holes cannot reach thermodynamic stability, due to the inevitable so-called Hawking radiation. In order to obtain a better understanding of the thermodynamic properties and phase transition of black holes, we must ensure that the black hole can achieve stability in the sense of thermodynamics. According to the previous results obtained by York et al.(1986) [21], achieving thermodynamic stability for asymptotic flat black hole system also depends on the effect of environments, i.e., one needs to consider the ensemble system. Different from the general thermodynamic system, the self-gravitational system has inhomogeneity in space, which makes it necessary to determine the corresponding thermodynamic quantities and their characteristic values.

The local thermodynamic stability of self-gravitation systems can be analyzed by considering the extreme value of the Helmholtz free energy of the system. When it reaches to a minimum value, the corresponding system is at least locally stable. According to the methods extensively studied in the literature [22–28], the system free energy ex-

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extreme value of gravitation systems can be derived from the action I , i.e., the partition function of the system at the zero-order approximation can be calculated by using the Gibbons-Hawking Euclidean action [29]

$$Z \approx e^{-I_E} \quad (1)$$

Combining this with the Helmholtz free energy F from the equation $Z = e^{-\beta F}$, we can obtain

$$I_E(r, T, Q; r_+) = \beta F = \beta E(r, Q; r_+) - S(r_+) \quad (2)$$

where r , T , Q respectively denote the radius, temperature and electric charge of the cavity, $\beta = 1/T$, and r_+ is the radius of the black hole horizon. $E(r, Q; r_+)$ and $S(r_+)$ are the internal energy and entropy of the black hole in the cavity. Therefore, when the r, T, Q is determined, the only variable for the system is r_+ . The thermal-equilibrium conditions of the black hole and environment can be determined by the following equation

$$\left. \frac{dI_E}{dr_+} \right|_{r_+=\bar{r}_+} = 0 \quad (3)$$

The conditions that the free energy reaches to its minimum value is that the system is at local equilibrium state. In order to reach to the thermal equilibrium it should satisfy the following criteria

$$\left. \frac{d^2 I_E}{dr_+^2} \right|_{r_+=\bar{r}_+} > 0. \quad (4)$$

Utilizing the method described above, the literature [23–29] have studied the charged black hole and black branes, obtained the requirements to meet the thermodynamic equilibrium conditions, and discussed the phase transition and critical phenomena. More recently, Eune et al. (2015) investigated the phase transition based on the corrections of Schwarzschild black hole radiation temperature [30]. More specifically, both of the charged black hole and the radiation field outside the black hole vision were considered in their work, in which the phase transition and critical phenomena of the system were discussed, under the condition that they are both in the equilibrium state.

On the other hand, Reissner-Nordstrom (RN) black hole and RN-AdS black hole are the exact solutions of the Einstein equation. The main difference between AdS space-time and asymptotically flat space-time is the famous Hawking-Page phase transition [1], i.e., the AdS background provides a natural constraint box, which makes it possible to form a heat stable between large stable black hole and hot gas. In the recent study of the phase transition the RN black hole, in order to meet the requirement of thermodynamic stability, one needs to artificially add a cavity concentric in the horizon of the black hole. However, the determination of the specific value of the radius of the cavity is still to be done. In the previous studies of phase transition of RN and RN-AdS black holes, it was found that both of the two types of black holes exhibit the same $P - V$ (fluid-gas phase transition) properties as van

der Waals-Maxwell system, which is also consistent with our finding through the comparison between the phase transition curves of these two kinds of black holes. Therefore, the consistency of the thermodynamic stable phase and phase transition between the cavity asymptotic flat black hole and the black hole in the AdS space seems to indicate a more profound connotation [22, 23]. One of the roles played by the cavity and the AdS space is to ensure the conservation of the degree of freedom within a certain system. Naturally, the discussion of the following problems is the main motivation of our analysis: Does a duality of a gravitational theory and a non-gravitational theory exist in the cavity setting? Are these two types of black holes inherently connected? If so, can the cavity radius introduced by the RN black hole phase transition be determined by the thermodynamic properties of the RN-AdS black hole?

In this paper, by comparing the phase transition curves of the RN black hole with the RN-AdS black hole, we establish the equivalent thermodynamic relations of the two kinds of black holes. We also discuss the relationship between the radius of cavity introduced into the RN black hole, the black hole horizon radius, and the cosmological constant. Finally, we investigate the equivalent thermodynamic quantities of two kinds of black holes, which provides theoretical basis for the further exploration of their internal relation.

2 Thermodynamic properties of flat space-time charged black hole

Reviewing the thermodynamic properties of RN black holes, the metric of a charged RN black hole is given by

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_2^2 \quad (5)$$

where

$$V(r) = 1 - \frac{2M}{r} + \frac{\bar{Q}^2}{r^2}. \quad (6)$$

The corresponding action expresses as [22, 26]

$$\begin{aligned} I_E(r_B, T_B, \bar{Q}; r_+) &= \beta_B F = \beta_B E(r_B, \bar{Q}; r_+) - S(r_+) \\ &= \beta_B r_B \left(1 - \sqrt{\left(1 - \frac{r_+}{r_B}\right) \left(1 - \frac{\bar{Q}^2}{r_B r_+}\right)} \right) - \pi r_+^2, \end{aligned} \quad (7)$$

Here r_B is the radius of concentricity outside a black hole horizon r_+ , which can be obtained from the following equation

$$V(r_+) = 1 - \frac{2M}{r_+} + \frac{\bar{Q}^2}{r_+^2} = 0 \quad (8)$$

The corresponding reduction are defined as

$$\bar{I}_E = \frac{I_E}{4\pi r_B^2}, x = \frac{r_+}{r_B}, q = \frac{\bar{Q}}{r_B}, \bar{b} = \frac{\beta_B}{4\pi r_B} \quad (9)$$

Note that the relation of $q < x < 1$ always holds (given $r_+ > \bar{Q}$, $r_B > r_+$) and the reduced action takes the form as

$$\bar{I}_E(\bar{b}, q, : x) = \bar{b} \left(1 - \sqrt{(1-x) \left(1 - \frac{q^2}{x} \right)} \right) - \frac{1}{4} x^2. \quad (10)$$

Therefore, we can obtain

$$\frac{d\bar{I}_x}{dx} = \frac{1 - \frac{q^2}{x^2}}{2(1-x)^{1/2} \left(1 - \frac{q^2}{x} \right)^{1/2}} (\bar{b} - b_q(x)), \quad (11)$$

where the function of the reciprocal of temperature is

$$b_q(x) = \frac{x(1-x)^{1/2} \left(1 - \frac{q^2}{x} \right)^{1/2}}{1 - \frac{q^2}{x^2}}. \quad (12)$$

and the reciprocal of the black hole radiation temperature can be written as

$$\frac{1}{T_2} = 4\pi r_B b_q(x) = 4\pi r_B \frac{x(1-x)^{1/2} \left(1 - \frac{q^2}{x} \right)^{1/2}}{1 - \frac{q^2}{x^2}}. \quad (13)$$

The condition for the RN Black hole and cavity to reach thermal equilibrium is

$$\frac{d\bar{I}_E}{dx} = 0 \quad \Rightarrow \quad \bar{b} = b_q(\bar{x}) \quad (14)$$

The critical charge and critical radius of a black hole can be determined by the following conditions

$$\left. \frac{db_q(x)}{dx} \right|_{x=x_c} = 0 \quad \left. \frac{d^2 b_q(x)}{dx^2} \right|_{x=x_c} = 0 \quad (15)$$

Combing Eqs. (13) and (15), one can easily derive the critical charge and the critical radius of black hole as [22]

$$q_c = \sqrt{5} - 2 \quad x_c = 5 - 2\sqrt{5} \quad (16)$$

The characteristic behavior curve of $b_q(x)$ to x is shown in Fig. 1.

3 $P - V$ criticality of charged AdS black holes

To start with, we review some basic thermodynamic properties of the spherical RN-AdS black holes. In the framework of Schwarzschild-like coordinates, the metric and the $U(1)$ field read

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2 \quad (17)$$

$$F = dA, \quad A = -\frac{Q}{r} dt \quad (18)$$

where $d\Omega_2^2$ stands for the standard element on s^2 and the function f is given by

$$f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{l^2}. \quad (19)$$

The radius of the black hole horizon, r_+ , satisfies the following equation

$$f(r_+) = 1 - \frac{2M}{r_+} + \frac{Q^2}{r_+^2} + \frac{r_+^2}{l^2} = 0 \quad (20)$$

and the black hole radiation temperature reads

$$T_1 = \frac{1}{4\pi r_+} \left(1 + \frac{3r_+^2}{l^2} - \frac{Q^2}{r_+^2} \right). \quad (21)$$

In order to explicitly illustrate the critical phenomenon of RN black hole, we present the $\beta - r_+$ diagram in Fig. 2, from which the phase-transformation point of system could be obtained as

$$Q_c = \frac{l}{6}, \quad r_c = \frac{l}{\sqrt{6}}, \quad \beta_c = \frac{\pi l \sqrt{6}}{2}. \quad (22)$$

These results are well consistent with those obtained in the previous analysis [2].

Moreover, the state parameters of a certain system should satisfy the first law of thermodynamics

$$dM = T_1 dS + \Phi dQ + V dP \quad (23)$$

where the potential Φ , thermodynamic volume V and pressure P of the black hole respectively express as

$$\Phi = \frac{Q}{r_+}, \quad V = \frac{4}{3} \pi r_+^3, \quad P = \frac{3}{8\pi l^2}. \quad (24)$$

4 Thermodynamic relationship of two space-time black holes

As can be seen from Fig. 1 and 2, for the two types of black holes, the change curves of the reciprocal of temperature with radius are quite similar to that for the Van der Waals-Maxwell gas-liquid phase transition (the corresponding critical index can be calculated at the critical point). Moreover, in the Ads space and ds space, the specific heat capacity of asymptotically flat black hole both expresses as $c_v \sim (T - T_c)^{-2/3}$ [26]. These common thermodynamic properties hint us the possibility to make the thermodynamic properties of the black hole in the space-time well consistent with each other, only by adjusting some specific parameters. Therefore, a cavity radius, which may ensure the thermodynamic equilibrium stability of a system, is introduced in our discussion of the thermodynamic properties of charged black holes in the flat space-time. Meanwhile, there is no definite value for the radius in our analysis, which provides us the possibility to adjust this parameter to obtain similar thermodynamic properties from the black holes in two different

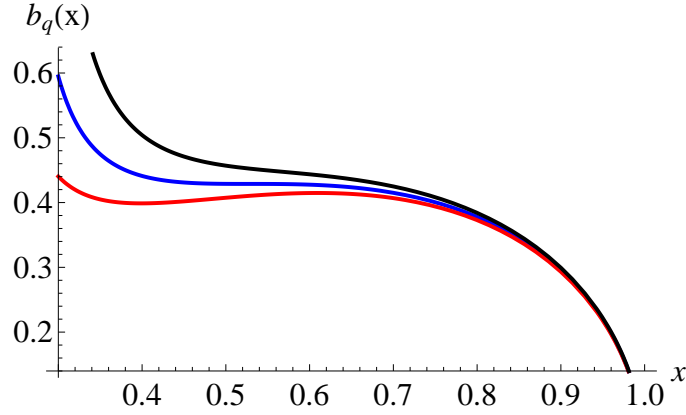


Fig. 1. The characteristic behavior of $b_q(x)$ as a function of x , when $q = q_c - \Delta q$ (red), $q = q_c$ (blue), $q = q_c + \Delta q$ (black). The blue curve in the middle represent the critical curve.

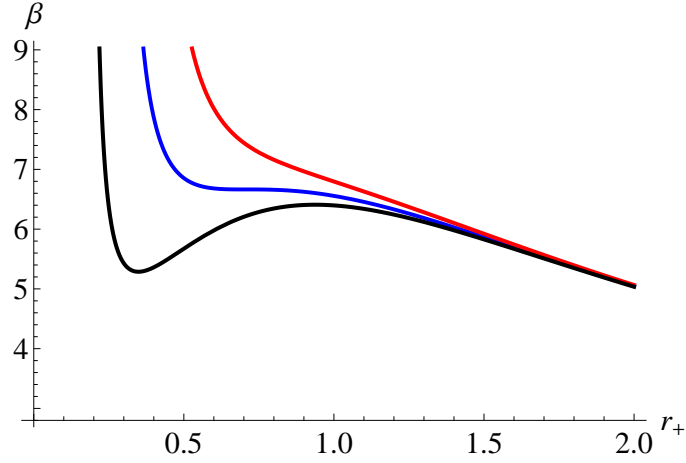


Fig. 2. $\beta - r_+$ diagram when $Q = Q_c - \Delta Q$ (red), $Q = Q_c$ (blue), $Q = Q_c + \Delta Q$ (black). The blue curve in the middle represent the critical curve.

space-time, and finally derive the relationship between the radius of the cavity and the radius of the black hole r_+ or the cosmological constant l .

It is generally believed that, if the corresponding state parameters in two thermodynamic system behave the same as the selected independent variables, these two systems will have the same thermodynamic properties. In order to obtain similar thermodynamic properties of the black holes in two space-time, we ensure that with the change of horizon radius of black hole, the temperature and entropy of the two systems behavior are equal. That is, the radii of the two black holes are equal to each other in two space-time, which leads to the similar entropy in the two systems. In the vicinity of the critical point, the temperature of a black hole in the space-time varies synchronously with the radius of the horizon. Based on the assumption that the entropy and temperature of a black hole in two space-time change the same with the horizon radius, we can obtain the relationship between the radius of the cavity introduced in the flat space-time and the radius of the black hole horizon and the cosmological constant.

Supposing the radii of the black hole horizon in the two space-time are both r_+ , which can be respectively determined by Eq. (8) and Eq. (20) for RN black holes and RN-AdS black holes. The common entropy of these two systems expresses as

$$S = \pi r_+^2 \quad (25)$$

As can be seen from the analysis in Section. 3, both of the two types of black holes show the same characteristics as Van der Waals-Maxwell system (fluid-gas phase transition), the critical points of which are respectively given by Eqs. (16) and (22). It is apparent that, Eq. (25) implies the critical point of RN black holes and RN-AdS black holes are determined when the black hole radius is the same in two space-time, while Eq. (22) shows the critical value of the RN-AdS black hole is dependent on the cosmological constant l . Therefore, when the horizon position r_+ and charge Q are given for such black hole, the radiation temperature depends on the cosmological constant. On the other hand, it can be seen from Eq. (13) that, when q and

x are fixed, the temperature of RN black holes depends on the cavity radius r_B .

To start with, when the black hole horizon r_+ is the same in the two space-time, we should determine the requirement of coincidence of critical curves, which is also the requirement to make $T_1 = T_2$. With the definition of $y = r_+/l$, the combination of Eq. (13) and (21) gives

$$1 - \frac{1}{36y^2} + 3y^2 = \frac{1 - q_c^2/x^2}{\sqrt{1-x}\sqrt{1-\frac{q_c^2}{x}}} \quad (26)$$

to obtain equal temperature in two time-space, where $Q_c = \frac{l}{6}$, $q_c = \sqrt{5}-2$. Because the black hole horizon position r_+ is the same for both types of black holes, the $x-y$ curve also qualifies the relation between l and r_B . We can get $y = y_c = 0.377039$ when $x = x_c = 5 - 2\sqrt{5}$, with which the critical curves $\beta(x) - x$ of RN black hole and $\beta(y) - x$ of RN-AdS black hole coincide the critical curve of Black Hole.

Secondly, at the critical point $y = y_c, x = x_c$, supposing there is a deviation of the charge to the threshold, we can get $\Delta Q = l\Delta\tilde{Q}$ and Δq from the following expression

$$\left(1 + 3\frac{r_+^2}{l^2} - \frac{Q^2}{r_+^2}\right) = \frac{1 - q^2/x^2}{\sqrt{1-x}\sqrt{1-\frac{q^2}{x}}} \quad (27)$$

It is straightforward to obtain

$$\begin{aligned} -\frac{2Q_c\Delta Q}{r_+^2} &= -\frac{\Delta\tilde{Q}}{3y_c^2} \\ &= \frac{1}{\sqrt{1-x_c}\left(1-\frac{q_c^2}{x_c}\right)^{3/2}} \frac{q_c}{x_c} \left(1 - \frac{2}{x_c} + \frac{q_c^2}{x_c^2}\right) \Delta q \end{aligned} \quad (28)$$

Substituting $y_c = 0.377039$, $x_c = 5 - 2\sqrt{5}$ into the above equation, we can get the relationship between $\Delta\tilde{Q}$ and Δq and the Eq. (27) can be rewritten as

$$\left(1 + 3y^2 - \frac{1}{36y^2} - \frac{\Delta\tilde{Q}}{3y^2}\right) = \frac{1 - q^2/x^2}{\sqrt{1-x}\sqrt{1-\frac{q^2}{x}}} \quad (29)$$

When $\Delta\tilde{Q}$ is given, one can get the value of $\Delta\tilde{q}$ from Eq. (28). Denoting $q = q_c + \Delta q = \sqrt{5}-2 + \Delta q$ in Eq. (29), we can obtain the $y-x$ curve deviating from the critical charge. The results are plotted in Fig. 4, in which the blue, red and black lines respectively correspond to the three cases with the critical charge, above the critical charge and under critical charge. It is obvious that when the relation between y and x satisfies Eqs. (26) and (29), the critical curves $\beta(x) - x$ of RN black holes and $\beta(y) - y$ of RN-AdS black holes coincide, which implies the equal temperature of the two space-time. Therefore, similar to the definition of entropy in Eq. (25), the heat capacity of the two systems can also be written as

$$C_Q = T_1 \left(\frac{\partial S_1}{\partial T_1} \right)_Q = C_{\tilde{Q}} = T_2 \left(\frac{\partial S_2}{\partial T_2} \right)_{\tilde{Q}} \quad (30)$$

We remark here that in the discussion above, the cavity radius is introduced as a state parameter to test the thermodynamic properties near the critical point of asymptotic flat space-time RN black hole. This procedure follows the study of the thermodynamic properties near the critical point of the RN-AdS black hole, where the cosmological constant is taken as the state parameter in the thermodynamic system. Our results demonstrates that the change of the two systems is the same near the critical point, as long as the variables y and x introduced in the two system satisfy Eqs. (26) and (29). As is shown in Figs. 1-2, the radius of the cavity is a one-to-one correspondence with the cosmological constant, and thus could be the dual of the cosmological constant in non-asymptotic flat space-time.

5 Conclusion

It is well known that black hole is an ideal system to understand the nature and behavior of quantum gravity. On one hand, black hole provides an ideal model to study all kinds of interesting behaviors of classical gravitation (under the sense of general relativity); on the other hand, it can be regarded as a macroscopic quantum system with unique thermodynamic properties (the entropy, temperature and holographic properties of gravitation are quantum), which provides an important window to probe the quantum gravity. More importantly, a better understanding of the black hole singularity, cosmological singularity, and cosmological inflation needs a basic theory of space-time gravitation. Up to now there is still no mature theory to precisely describe the quantum characteristics of the asymptotic flat black holes and non-asymptotic flat space-time black holes. However, string theory firstly provided the microcosmic explanations for the entropy of some AdS black holes, which predicted the existence of M-theory and some duality relations (especially AdS / CFT correspondence) and realized the holographic properties of gravitational systems [32].

In this paper, by setting the black-hole horizon at the same value and assuming the temperature near the critical point is equal in two space-time, we have discussed the relationship between the radius of the cavity r_B and the cosmological constant l . The former parameter is always introduced in studying the thermodynamics of the charged black hole in the flat space-time, while the latter is related to the non-flat space-time. Our results may provide a theoretical basis for exploring the internal relations of thermodynamics in the black space. Moreover, by establishing the correspondence between the thermodynamics parameters of the asymptotic flat space-time and non-asymptotic flat space-time, we hope to seek the correspondence between the quantum properties of the black hole in two space-time, which is beneficial to understand the thermodynamics and quantum properties of black holes in different space-time.

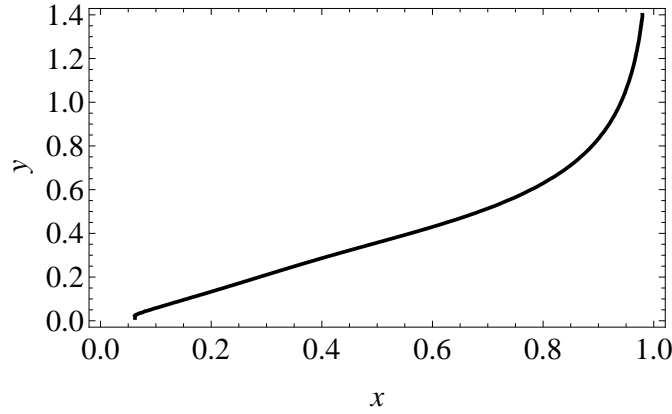


Fig. 3. The $y - x$ curve for Eq. (26) with $Q_c = \frac{l}{6}$, $q_c = \sqrt{5} - 2$, $l = \sqrt{3}$.

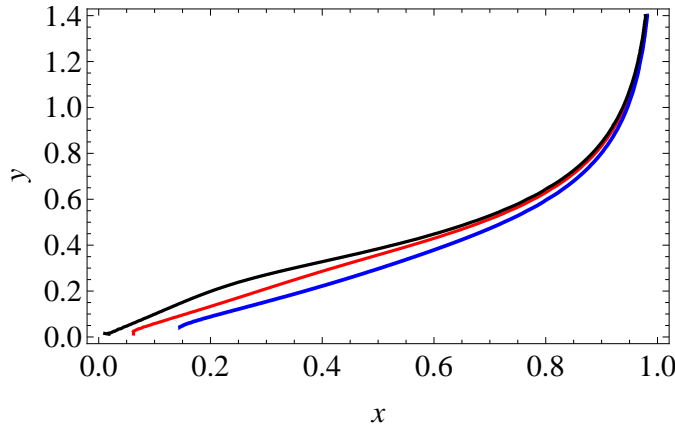


Fig. 4. The $y - x$ curves for $\Delta\tilde{Q} = 0.05$ (blue), $\Delta\tilde{Q} = 0$ (red), and $\Delta\tilde{Q} = -0.05$ (black).

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